Problem set 4

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Let G be a group, and H be a subgroup. The subgroup H is called **normal** if for any $g \in G$ we have $gHg^{-1} = H$ (equality of sets!). In other words, H is normal if and only if all left cosets are the same as right cosets, $gH = Hg$.
Problem 1. For a group G prove the following.

1. If G is abelian, every its subgroup is normal.
2. The two subgroups $\{e\}$ and G itself are both normal.
3. The **center** $Z(G) = \{g \in G \mid \forall h \in G, hg = gh\}$ is a normal subgroup.

Problem 2. The group D_{2n} of symmetries of a regular n-gon has normal subgroup $\mathbb{Z}/n$ consisting of rotations.
Problem 3. The subgroup A_n of even permutations is normal in S_n.
Problem 4. For any (finite) group G and a subgroup H of index 2, H is necessarily normal in G.

The point is: if H is normal, the set of cosets G/H has a natural group structure. This group is called the **quotient group**. We define $aH * bH := abH$. Why is it well-defined?

The next three exercises show that normal subgroups are essentially the same as kernels of homomorphisms. In other words, for a group G there is a bijection between the set of normal subgroups $H \subset G$ and **surjective** homomorphisms $G \twoheadrightarrow K$.

Problem 5. For any homomorphism $\varphi\colon G \to K$, $\ker\varphi$ is a **normal** subgroup.
Problem 6. If $\varphi\colon G \to K$ is a **surjective** homomorphism, then $K \simeq G/\ker\varphi$.
Problem 7. The First Isomorphism Theorem. For **any** homomorphism $\varphi\colon G \to K$, there is an isomorphism $G/\ker\varphi \simeq \operatorname{im}\varphi$.

Therefore, the image of any homomorphism really “looks like” a quotient group. (Compare with a similar result about the orbits of group actions we have seen before.)

Problem 8. Let $n\mathbb{Z} \subset \mathbb{Z}$ be the subgroup $\{\dots, -n, 0, n, 2n, \dots\}$. Prove that $\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/n$ (Hint: use the First Isomorphism Theorem).
Problem 9. Prove that $S_n/A_n \simeq \mathbb{Z}/2$.
Problem 10. Prove that $G \times H/H \simeq G$, where $H \subset G \times H$ is the subgroup $H = \{(e,h) \mid h \in H\}$.

Let G be a group and K be another group, on which G acts by **automorphisms**. In other words, for each $g \in G$ we have assigned an isomorphism $A_g\colon K \to K$, such that $A_e = id$ and $A_{gh} = A_g \circ A_h$. We write ${}^g k$ (or $g.k$) for $A_g(k)$.
We define **semi-direct product** $G \ltimes K$ to be the set $K \times G$ with the operation

$$(k_1, g_1) * (k_2, g_2) = (k_1\, {}^{g_1}k_2, g_1 \cdot g_2)$$

Problem 11. Prove that $G \ltimes K$ is again a group, with $K = \{(k,e) \mid k \in K\}$ being a normal subgroup in $G \ltimes K$, and $G \ltimes K/K \simeq G$.
Problem 12. If G acts trivially on K, then $G \ltimes K \simeq G \times K$.
Problem 13. The group D_{2n} is isomorphic to $\mathbb{Z}/2 \ltimes \mathbb{Z}/n$ with the action of $1 \in \mathbb{Z}/2$ given by $a \mapsto -a$.
Problem 14. The group S_n is $\mathbb{Z}/2 \ltimes A_n$, where the action is by conjugation by any odd permutation.
Problem 15. The group of symmetries of a cube is isomorphic to the semi-direct product $S_3 \ltimes (\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2)$. (Hint: think of vertices of a cube as sequences $(\pm1, \pm1, \pm1)$.)
Try to see what these groups mean geometrically. Therefore, you get an isomorphism of groups

$$S_4 \times \mathbb{Z}/2 \simeq S_3 \ltimes (\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2)$$